13 Orthogonal Decomposition

Orthogonal Complement For a subset $\mathcal{M}$ of an inner-product space $\mathcal{V}$, the **orthogonal complement** $M^\perp$ (pronounced "$\mathcal{M}$ perp") of $\mathcal{M}$ is defined to be the set of all vectors in $\mathcal{V}$ that are orthogonal to every vector in $\mathcal{M}$. That is,

$$\mathcal{M}^\perp = \{\boldsymbol{x} \in \mathcal{V} \mid \langle \boldsymbol{m}, \boldsymbol{x} \rangle = 0 \text{ for all } \boldsymbol{m} \in \mathcal{M}\}.$$

1. For an inner-product space $\mathcal{V}$, what is $\mathcal{V}^\perp$? What is $\mathbf{0}^\perp$?

2. For every inner-product space $\mathcal{V}$, prove that if $M \subseteq V$, then $\mathcal{M}^\perp$ is a subspace of $\mathcal{V}$.

3. If $\mathcal{M}$ is a subspace of a finite-dimensional inner-product space $\mathcal{V}$, show that then $\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^\perp$.

4. If $\mathcal{N}$ is a subspace such that $\mathcal{V} = \mathcal{M} \oplus \mathcal{N}$ and $\mathcal{N} \perp \mathcal{M}$ (every vector in $\mathcal{N}$ is orthogonal to every vector in $\mathcal{M}$), show that then $\mathcal{N} = \mathcal{M}^\perp$.

Orthogonal Complementary Subspaces If $\mathcal{M}$ is a subspace of a finite-dimensional inner-product space $\mathcal{V}$, then

$$\mathcal{V} = \mathcal{M} \oplus \mathcal{M}^\perp. \tag{2}$$

Furthermore, if $\mathcal{N}$ is a subspace such that $\mathcal{V} = \mathcal{M} \oplus \mathcal{N}$ and $\mathcal{N} \perp \mathcal{M}$ (every vector in $\mathcal{N}$ is orthogonal to every vector in $\mathcal{M}$), then

$$\mathcal{N} = \mathcal{M}^\perp.$$

5. Let $U \in \mathrm{Mat}_{m\times m}(\mathbb{R})$, $U = (U_1|U_2)$ be a partitioned orthogonal matrix. Explain why $\mathrm{im}(U_1)$ and $\mathrm{im}(U_2)$ must be orthogonal complements of each other.

6. If $\mathcal{M}$ is a subspace of an n-dimensional inner-product space, show that then $\dim(\mathcal{M}^\perp) = n - \dim(\mathcal{M})$ and $\mathcal{M}^{\perp^\perp} = \mathcal{M}$.

Perp Operation If $\mathcal{M}$ is a subspace of an n-dimensional inner-product space, then the following statements are true.

- $\dim(\mathcal{M}^\perp) = n - \dim(\mathcal{M})$.
- $\mathcal{M}^{\perp^\perp} = \mathcal{M}$.

7. If $\mathcal{M}$ and $\mathcal{N}$ are subspaces of an n-dimensional inner-product space, prove that the following statements are true.

(a) $\mathcal{M} \subseteq \mathcal{N} \Longrightarrow \mathcal{N}^\perp \subseteq \mathcal{M}^\perp$.

(b) $(\mathcal{M} + \mathcal{N})^\perp = \mathcal{M}^\perp \cap \mathcal{N}^\perp$.

(c) $(\mathcal{M} \cap \mathcal{N})^\perp = \mathcal{M}^\perp + \mathcal{N}^\perp$.

8. For every $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ show that $\mathrm{im}(A)^\perp = \ker(A^\top)$ and $\ker(A)^\perp = \mathrm{im}(A^\top)$.

Orthogonal Decomposition Theorem For every $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$

$$\mathrm{im}(A)^\perp = \ker(A^\top) \quad \text{and} \quad \ker(A)^\perp = \mathrm{im}(A^\top).$$

In light of (2), this means that every matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ produces an orthogonal decomposition of $\mathbb{R}^m$ and $\mathbb{R}^n$ in the sense that

$$\mathbb{R}^m = \mathrm{im}(A) \oplus \mathrm{im}(A)^\perp = \mathrm{im}(A) \oplus \ker(A^\top),$$

and

$$\mathbb{R}^n = \ker(A) \oplus \ker(A)^\perp = \ker(A) \oplus \mathrm{im}(A^\top),$$

URV Factorization For each $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ of rank r, there are orthogonal matrices $U \in \mathrm{Mat}_{m\times m}(\mathbb{R})$ and $V \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and a nonsingular matrix $C \in \mathrm{Mat}_{r\times r}(\mathbb{R})$ such that

- The first r columns in U are an orthonormal basis for $\mathrm{im}(A)$.
- The last $m-r$ columns of U are an orthonormal basis for $\ker(A^\top)$.
- The first r columns in V are an orthonormal basis for $\mathrm{im}(A^\top)$.
- The last $n-r$ columns of V are an orthonormal basis for $\ker(A)$.

Each different collection of orthonormal bases for the four fundamental subspaces of A produces a different URV factorization of A. In the complex case, replace replace $(\star)^\top$ by $(\star)^*$ and "orthogonal" by "unitary".

9. Verify the orthogonal decomposition theorem for $A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}$.

10. Find a basis for the orthogonal complement of $\mathcal{M} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 6 \end{pmatrix} \right\}$.

11. Explain why the rank plus nullity theorem

$$\dim \mathrm{im}(A) + \dim \ker(A) = n \qquad \forall A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$$

is a corollary of the orthogonal decomposition theorem.

12. Compute a URV factorization for the matrix $A = \begin{pmatrix} -4 & -2 & -4 & -2 \\ 2 & -2 & 2 & 1 \\ -4 & 1 & -4 & -2 \end{pmatrix}$ by using elementary row operations together with Gram-Schmidt orthogonalization.

> **Range Perpendicular to Nullspace** For $\operatorname{rank}(A) = r$ ($A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$), the following statements are equivalent:
>
> - $\operatorname{im}(A) \perp \ker(A)$,
> - $\operatorname{im}(A) = \operatorname{im}(A^\top)$,
> - $\ker(A) = \ker(A^\top)$,
> - $A = U \begin{pmatrix} C_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} U^\top$
>
> in which U is orthogonal and C is nonsingular. Such matrices will be called **RPN matrices**, short for "range perpendicular to nullspace". Some authors call them *range-symmetric* or EP matrices. Nonsingular matrices are trivially RPN because they have a zero nullspace. For complex matrices, replace $(\star)^\top$ by $(\star)^*$ and "orthogonal" by "unitary".

13. Compute a URV factorization for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 5 \end{bmatrix}$.

14. Basis for a vector space $\mathcal{L} = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \,|\, x_1 + x_2 = 0, -x_1 + 2x_2 + x_3 = 0\}$ is $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\}$. Find orthogonal complement of $\mathcal{L}$ (with respect to standard inner product $\langle x, y\rangle = x^\top y$).

15. In inner product space $\mathcal{P}_2 = \{p(x) = ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ of all polynomials of degree less or equal 2 with inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,\mathrm{d}x$ let $\mathcal{M}$ be a given subspace defined with

$$\mathcal{M} = \operatorname{span}\{x^2 - 1, x + 1\}.$$

Find a basis for $\mathcal{M}^\perp$, and write a polynomial $p(x) = 2x^2 + x + 5$ in form $p = p_1 + p_2$, where $p_1 \in \mathcal{M}$, $p_2 \in \mathcal{M}^\perp$.

16. In inner product space $\mathbb{R}^4$, with inner product defined with

$$\langle x, y\rangle = x_1y_1 + 2x_2y_2 + x_3y_3 + 2x_4y_4$$

let $\mathcal{V}$ be a given subspace spanned with vectors $v_1 = (1, 0, 1, 0)^\top$ and $v_2 = (1, 0, 1, 1)^\top$. Write vector $x = (4, 2, 2, 4)^\top$ in form $x = v + w$, where $v \in \mathcal{V}$, $w \in \mathcal{V}^\perp$.

17. Let $\mathcal{M}$ denote subspace of inner product space $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ spanned with matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$. Find a basis for orthogonal complement of $\mathcal{M}$, and write the matrix $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in the form $X = Y_1 + Y_2$, where $Y_1 \in \mathcal{M}$, and $Y_2 \in \mathcal{M}^\perp$. (Standard inner product in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ is defined with $\langle A, B\rangle = \operatorname{trag}(AB^\top)$).

18. In inner product space $\mathcal{P}_3 = \{at^3 + bt^2 + ct + d \,|\, a, b, c, d \in \mathbb{R}\}$ of all polynomials of degree ≤ 3 with inner product defined with

$$\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)\,\mathrm{d}t$$

Let $\mathcal{M} = \operatorname{span}\{1 + t, 1\}$ be a given subspace. Find a basis for $\mathcal{M}^\perp$.

Alston Scott Householder[11]

[11] Alston Scott Householder (1904 – 1993) was one of the first people to appreciate and promote the use of elementary reflectors for numerical applications. For $\boldsymbol{u} \neq \mathbf{0}$ ($u \in \mathbb{C}^n$), the elementary reflector about $u^\perp$ is defined to be

$$R = I - 2\frac{\boldsymbol{u}\boldsymbol{u}^*}{\boldsymbol{u}^*\boldsymbol{u}}$$

or, equivalently,

$$R = I - 2\boldsymbol{u}\boldsymbol{u}^* \qquad \text{when} \qquad \|u\| = 1.$$

Elementary reflectors are also called Householder transformations. Although his 1937 Ph.D. dissertation at University of Chicago concerned the calculus of variations, Householder's passion was mathematical biology, and this was the thrust of his career until it was derailed by the war effort in 1944. Householder joined the Mathematics Division of Oak Ridge National Laboratory in 1946 and became its director in 1948. He stayed at Oak Ridge for the remainder of his career, and he became a leading figure in numerical analysis and matrix computations. Like his counterpart J. Wallace Givens at the Argonne National Laboratory, Householder was one of the early presidents of SIAM.